Polyhedral Single Degree-of-freedom Expanding Structures

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Abstract

Some engineering applications require structures to expand and contract in size, while retaining their exterior shape. The applications range from mundane daily life objects to more fancy art structures. In contrast to a multi degree-of-freedom structure, a single degree-of-freedom structure can be driven by a single actuator, reducing cost and simplifying the control. In this paper, we study single degree-of-freedom structures that can be formed by a lattice of single degree-of-freedom polyhedral expanding units. Due to built-in symmetries, the entire structure can expand and contract as one of the units in the structure is actuated.

1. Introduction

In recent years, Hoberman structures have gained popularity as "cool toys". A recent Hoberman's kit "Expandagon" allows a user to assemble expanding units to form fascinating single degree-of-freedom expanding shapes [1]. Each Hoberman unit is made using revolute joints. However, due to the nature of their design, the exterior shape does not remain the same during expansion. Other symmetric mechanisms such as Octoids and Fulleroids are special 3-dimensional structures that work due to careful design of the mechanism ([5], [2]). Another class of structures, referred to as "inflatable structures" has been extensively used in space. This technology uses fabric of special material which hardens to a desired shape of the structure. Modular robotics is a growing field today. Here, modules can connect and disconnect with other modules. This technology results in robots that can "metamorph" or change shape during operation [4], [6].

The unique contributions of this paper are: (i) Systematic guidelines to construct single degree-of-freedom expanding structures starting out from single degree-of-freedom polyhedral expandable units; (ii) Each polyhedral unit is made of prismatic joints, as opposed to revolute joints in Hoberman designs; (iii) The overall structure can expand and contract while retaining the exterior shape; (iv) A dynamic analysis is presented to facilitate design and selection of actuators; and (v) The procedures are illustrated by case study of a chair that can accommodate to the size of a user.

The organization of this paper is as follows: Section 2 describes Hoberman single degree-of-freedom polygonal structures. Section 3 describes the geometry of classes of polyhedra and construction of single degree-of-freedom polyhedral units. These polyhedral units are assembled into expandable single degree-of-freedom lattices outlined in Section 4. A systematic analysis of dynamic equations of a lattice is performed in Section 5. An example of an expandable design is discussed in Section 6.

2. Hoberman's Designs

Expanding Sphere and *Expandagon* [1] are designs that use revolute joints for interconnecting members of the mechanism. The underlying concept in these designs is an expanding regular "N-gon". Figs. 1 and 2 show designs based on regular 4-gon and 6-gon. Because of circular symmetry, expanding N-gons can be placed along great circles of a sphere through a point on its surface, such as the north pole of a sphere. Other great circles can then be tied to the existing N-gons to form other great circles. As a result, the mechanism as a whole expands once one of the planar components expands.

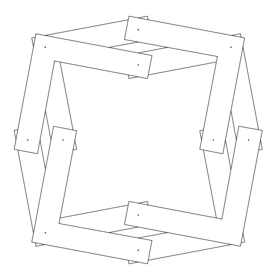

Fig. 1: A schematic of an expanding "4-gon" based on Hoberman sphere design.

From a design point of view, one needs to ensure that the adjacent units of an N-gon expand out in a symmetric manner. Consider a portion of an N-gon, as shown in Fig. 3, in which every line is a link and every intersection of two lines is a joint. For example, EBC and CAD are two successive "caps" that form the regular N-gon, and so are identical. Their cone angles $\angle$EBC and $\angle$CAD must be the same during expansion.

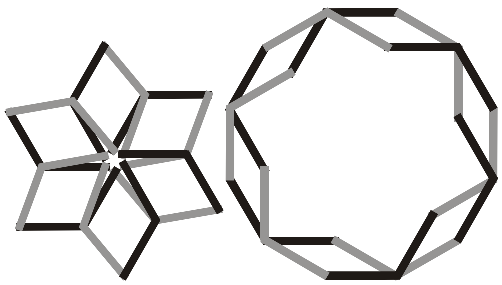

Fig. 2: A schematic of an expanding "6-gon" in the contracted and expanded state, based on Hoberman sphere design.

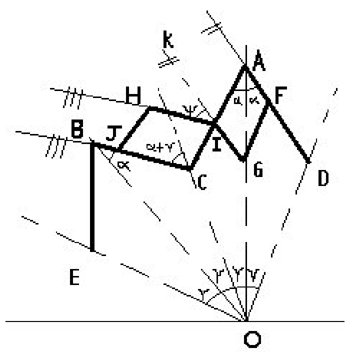

Fig. 3: The relationships between the joint angles of an N-gon Hoberman design.

In order to ensure this relationship, two parallelograms AFGI and CIHJ are made to transmit the motion from the unit CAD to the next unit EBC. This choice of the parallelograms is consistent with the "Expandagon" design by Hoberman. From the geometry in Fig. 3, one can show that the angles $\angle$EBC and $\angle$CAD are equal to 2α if the following two conditions are met: (i) HIG is a rigid member; and (ii) the angle $\angle$HIK=$\psi = 2\gamma$, where 2γ is the subtended cone angle at the center of the circle by a "cap" of the N-gon. For a 4-gon, $\gamma = \frac{\pi}{4}$, hence $\angle$HIK=$\frac{\pi}{2}$. Similarly, for a 6-gon, $\gamma = \frac{\pi}{6}$, hence $\angle$HIK=$\frac{\pi}{3}$.

In a Hoberman Expandagon design, a single symmetrically expanding N-gon requires 4N members that are connected by 6N revolute joints. Hence, the number of moving members and joints become quickly large as the number N increases. In *Expandagon*, the planar expanding polygons are interconnected by two degree-of-freedom joints to obtain 3-dimensional expanding structures. This kit, while extremely fascinating has two limitations: (i) During expansion, the design changes its overall exterior shape; (ii) Because of planar modules used in the construction, a polyhedral unit has a large number of moving members.

As an alternative, in this paper, we consider expanding structures that use only prismatic joints instead of revolute joints. Also, we attempt in this paper is to systematically describe a procedure to create 3-dimensional expanding structures motivated by geometry of regular polyhedral shapes.

3. Single Dof Polyhedral Units

A large number of polyhedra can be constructed. However, only five have the following two properties: (i) all faces of the polyhedra are identical; (ii) the same number of faces and edges meet at each vertex. Such special polyhedra are also called "regular polyhedra" [3]. These are: (I) Tetrahedron, (ii) Octahedron, (iii) Cube, (iv) Icosahedron, (v) Pentagonal dodecahedron. A "semi-regular polyhedra" is one which has the same number of edges meeting at every vertex but different kinds of faces. Some examples of semi-regular polyhedra are cuboctahedron and icosadodecahedron. A cuboctahedron is formed by slicing the corners of a cube or an octahedron, hence the name. Similarly, icosadodecahedron results by slicing the corners of an icosahedron or a pentagonal dodecahedron. In both these examples, each vertex has the same number of edges but not all faces are the same. A large number of polyhedra exist which do not fall into the above two categories [3].

In this study, we focus on single degree-of-freedom expandable units that can be constructed using regular polyhedra. The choice of regular polyhedra is motivated from their built-in symmetries. The expanding units are created by replacing an edge of a polyhedron by a prismatic joint, while retaining the relative orientation between the edges of the polyhedron. This concept is sketched in Fig. 4, where the edges of a tetrahedron are replaced by prismatic joints. Except for the cube and pentagonal dodecahedron, it can be shown that such expandable units possess a single internal degree-of-freedom, i.e., any one of the prismatic joints can be lengthened or shortened and the regular polyhedra will remain the same.

A simple argument for expansion of a regular tetrahedron is as follows: If the three faces of a regular tetrahedron are opened up from their common joining vertex as shown in Fig. 5, one gets three equilateral triangles in addition to the base. On actuating any one of the edges, the four triangles expand out equally. They can be assembled back as a regular tetrahedron.

Fig. 4 shows a photograph of an expandable tetrahedral unit built at University of Delaware in two different configurations.

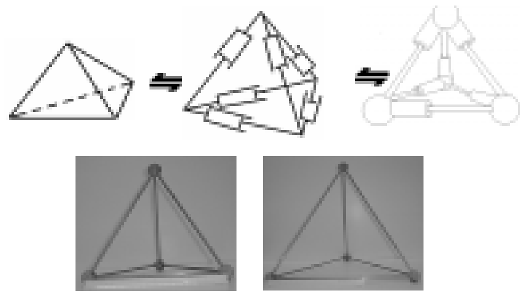

Fig. 4: The edges of a regular tetrahedron are replaced by prismatic joints to form a single degree-of-freedom expandable unit. A photograph of a tetrahedral unit fabricated at University of Delaware.

A similar reasoning can be given for the expanding octahedron and icosahedron by opening these in two dimensional net diagrams, as shown in Fig. 5 [3]. From the two net diagrams, it is clear that the two expanding units have a single degree-of-freedom because if any one of the equilateral triangles are expanded, the entire unit expands by the same amount in order to maintain the same angles.

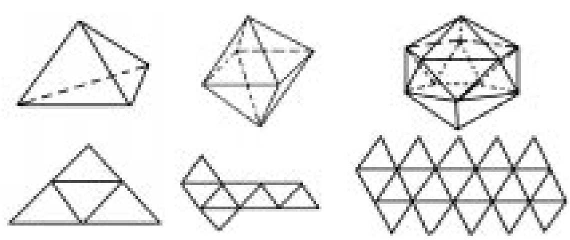

Fig. 5: The net diagrams for the octahedron and icosahedron show that the systems have single degree-of-freedom.

If all the edges of a cube are replaced by prismatic joints, the overall system has three degrees-of-freedom since many of the edges are parallel. These degrees-of-freedom are the displacements along three orthogonal directions. However, if two additional prismatic joints are added to the two adjacent face diagonals, as shown in Fig. 6, the system becomes single degree-of-freedom. This design requires fourteen prismatic joints. A similar design is shown for the pentagonal dodecahedron, where the pentagonal faces have been triangulated.

For a cube, an alternative is to use the six face diagonals to form an inscribed tetrahedron inside the cube and actuate the tetrahedron. This actuation concept is developed further in the next section and is demonstrated through an example in Section 6.

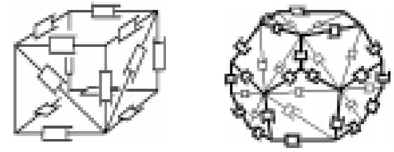

Fig. 6: Single degree-of-freedom cube actuated by the edges or the inscribing tetrahedron. A pentagonal dodecahedron module actuated.

4. Single DOF Polyhedral Lattices

There are three different ways of interconnecting polyhedral units: (i) vertex to vertex joining; (ii) edge to edge joining; and (iii) face to face joining. In all of these cases, the joining means the components of the polyhedral units are rigidly attached together. Thus no rotations are allowed anywhere within the whole system, only prismatic translations.

On joining polyhedra at the vertices, one can produce some uniform and well-defined structures. For example, four tetrahedral units are joined at the vertices to form a larger tetrahedron in Fig. 7. In an edge joining lattice, as the name suggests, the neighboring polyhedra share a common edge. This figure shows an example where two cubes are connected at their common edge. Polyhedra can also be joined at the faces, as shown in the cubic lattice of Fig. 7.

A single degree-of-freedom lattice can be formed using single degree-of-freedom polyhedral building blocks by combining them with one of the methods mentioned above as long as at least one prismatic joint from one block has its motions constrained to follow the motion of one on the joining block (i.e. for the vertex joining, more than one vertex on a block needs to be joined). The objective of the design then is to actuate the lattice using just one input, anywhere in the mechanism. Expansion of any one polyhedral unit enables the entire lattice to expand, while retaining the exterior shape.

Fig. 8 shows schematically how cubic units can be used to construct an array, which are then stacked on each other to provide a richer 3-dimensional array. Such a lattice shares common edges and faces. In the cubic lattice, diagonal edges are required only on two sides to make the entire lattice reduce to a single degree-of-freedom. Fig. 8 also shows an interconnection of cubic units where the inscribed tetrahedron is used for actuation.

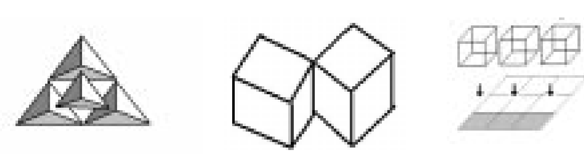

Fig. 7: Vertex, edge and face joining modules.

Sometimes, interconnections of regular polyhedral units may give rise to patterns with inter-polyhedral voids. For example, if a tetrahedral lattice is arranged vertex-to-vertex in an array, the inter-unit spaces are octahedral. Alternate tetrahedron and octahedron can give rise to space filling lattice.

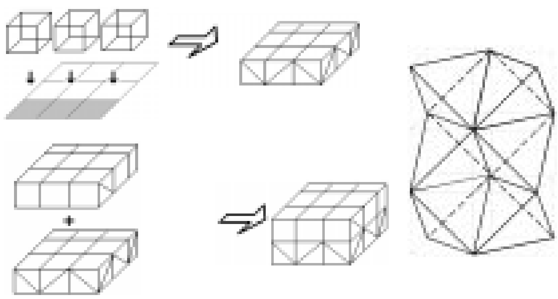

Fig 8: A 3-dimensional lattice formed by cubic modules. The actuation could be either through the edges or through the edges of an inscribed tetrahedron.

Polyhedral lattices can also be made by arranging them symmetrically within imaginary cubes. If these imaginary cubes are close-packed, the enclosed polyhedra can share a face or an edge. For example, Fig. 9 shows an icosahedron placed inside a cube so that an edge of the icosahedron touches each face of the cube. Then, a series of imaginary cubes, each containing a similar icosahedron, can share a face. Pentagonal dodecahedra can also be arranged similarly inside cubes as shown.

Instead of single polyhedron, groups of polyhedra can also be arranged inside a single imaginary cube, with each cube touching the face of other cubes arranged symmetrically [3].

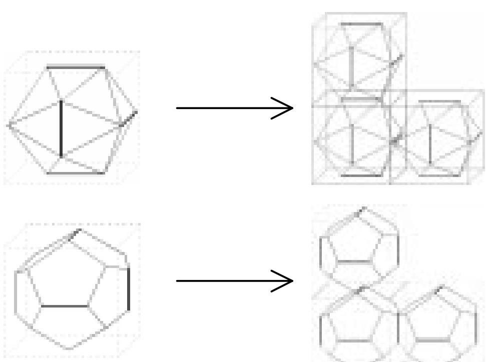

Fig. 9: An icosahedron/dodecahedron placed inside a cube so that an edge touches each face of the cube.

5. Dynamic Analysis of Lattices

From the perspective of actuator selection of such polyhedral lattices, it is important to perform a dynamic analysis. The dynamic analysis requires knowledge of kinematics of the mechanism as well as inertia distribution in the lattice. An interesting feature of the kinematics of a polyhedral mechanism is that each moving member maintains the same orientation with respect to all other members in the mechanism, i.e., each member is in pure translation with respect to the inertial frame if one of the members is grounded.

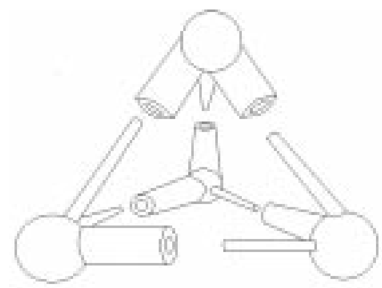

Fig 10: A tetrahedron is a set of four nodes, each consisting of three prongs which slide respect to adjacent nodes.

A lattice is made up of nodes with fixed members that slide in and out with respect to members of the adjoining nodes. For example, as shown in Fig. 10, a tetrahedron is a set of four nodes, each consisting of three prongs that slide with respect to the prongs of the adjacent nodes. Hence, the number of rigid bodies in a lattice is the same as number of nodes. Also, to simplify the analysis we assume that the mass of an element is concentrated at the nodes. In some lattices designed for a specific application, some nodes and edges may be missing.

As described earlier, a tetrahedral lattice can also be viewed as an octahedral lattice because vertex to vertex connecting tetrahedral units leave octahedral gaps in between. A systematic method is presented here to characterize a tetrahedral/cubic lattice.

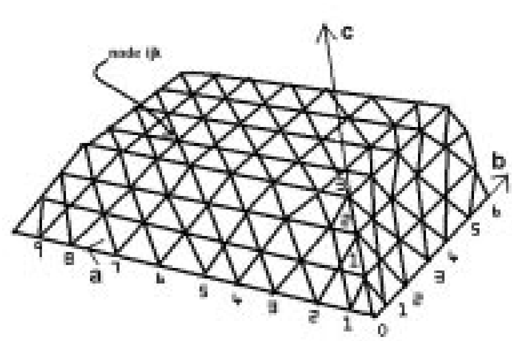

Fig. 11: The coordinate axes to identify a node on a tetrahedral lattice.

In a lattice shown in Fig. 11, the nodes must be numbered in an orderly fashion. Hence, we define a set of three coordinate axes **a**, **b**, **c** with unit vectors at $\frac{\pi}{3}$ with each another in a tetrahedral lattice and at $\frac{\pi}{2}$ in a cubic lattice. Without loss of generality, we consider the (0,0,0) node to be inertially fixed. Each node in the lattice can be described uniquely by an integer triple (i,j,k) with inter-nodal distance of l along **a**, **b**, **c**. The location of a node with respect to fixed origin is

$$\mathbf{p}_{ijk} = l(i\mathbf{a} + j\mathbf{b} + k\mathbf{c}) \qquad (1)$$

where **a**, **b**, **c** are respectively the unit vectors and *i, j, k* are integers. On differentiation, the inertial velocity of the node is

$$\mathbf{V}_{ijk} = \dot{l}(i\mathbf{a} + j\mathbf{b} + k\mathbf{c}) = \tilde{\mathbf{V}}_{ijk}\dot{l}\,. \qquad (2)$$

The contribution of the node (i,j,k) to the kinetic energy of the system is

$$K_{ijk} = \left(\frac{1}{2}\right) m_{ijk} \left(\tilde{\mathbf{V}}_{ijk} \cdot \tilde{\mathbf{V}}_{ijk}\right)\dot{l}^2 \quad . \qquad (3)$$

The contribution of the node (i,j,k) to the potential energy of the system is

$$P_{ijk} = m_{ijk}\, g\, \mathbf{p}_{ijk}\, .\mathbf{n} \;, \qquad (4)$$

where g is the gravity constant and **n** is a vector opposite in direction to the gravity vector.

On using Lagrange's equations, the resulting dynamic equations for the polyhedral lattice are

$$\sum_i \sum_j \sum_k m_{ijk} \left(\tilde{\mathbf{V}}_{ijk} \cdot \tilde{\mathbf{V}}_{ijk}\right)\ddot{l} + m_{ijk}\, g\mathbf{p}_{ijk}\, .\mathbf{n} = u \qquad (5)$$

On substituting the velocity of the nodes, the dynamic equations of motion for a tetrahedral lattice are

$$\sum_i \sum_j \sum_k m_{ijk}\,(i^2 + j^2 + k^2 + ij + jk + ki)\ddot{l} + \alpha m_{ijk}\, gk) = u, \qquad (6)$$

where $\alpha = \mathbf{c}\cdot\mathbf{n}$ and for a cubic lattice are

$$\sum_i \sum_j \sum_k [m_{ijk}\,(i^2 + j^2 + k^2)\ddot{l} + m_{ijk}\, gk] = u. \qquad (7)$$

These equations give the actuator force required to expand and contract the lattice. As mentioned earlier, some of the nodes or edges in the lattice could be missing. In these situations, the mass of the node (i,j,k) must be suitably modified.

6. Chair Example

In this section, we study of a lattice that approximates a chair as shown in Fig. 12.

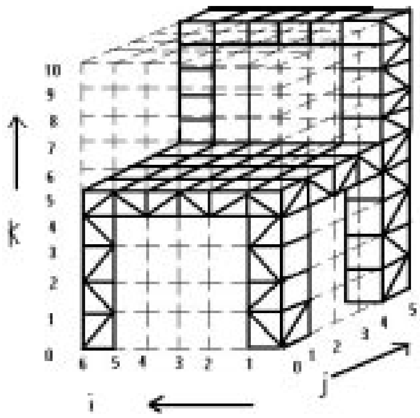

Fig. 12: A lattice that approximates a chair. The diagonal members on the side face ensure propagation of equal expansion along the three mutually orthogonal directions of the cubes forming the lattice.

Mass Matrix(m_{ijk}) j=0							
i k	0	1	2	3	4	5	6
10	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
5	3	6	4	6	4	6	3
4	7	5	6	4	6	5	6
3	4	6	0	0	0	6	4
2	6	4	0	0	0	4	6
1	4	5	0	0	0	5	4
0	5	3	0	0	0	3	5

Table 1: The j=0 nodal masses for the chair shown in Fig. 12. It is assumed that the mass of a node is directly proportional to the number of edge connections of a node.

From the geometry of the chair and edge interconnections between the nodes, one can write the mass for each node *i,j,k* in the lattice. For example, the nodal masses for j=0 are shown in Table 1. If one focuses on the nodes with k values between 6 and 10, none have any interconnection, therefore, any mass. For k=5, i=1 has three interconnections, hence its nodal mass is three units. Similarly, i=2 has six interconnections, therefore, a mass of 6 and so on. The equations of motion for the chair using cubic lattice according to Eq.(7) are:

$$45816m\ddot{l} + 3944mg = u \qquad (8)$$

As described in Fig. 8, the same design of the chair could also be accomplished by inscribing a tetrahedron within each cube forming the lattice. This alternative design is shown in Fig. 13. This lattice has only six prismatic joints within each cube, as opposed to twelve (or more) in the design of Fig. 12. Similar to Table 1, nodal masses can be developed for the design of Fig. 13. For j=0, these nodal masses are shown in Table 2. The equations of motion for this tetrahedral lattice of the chair are as follows:

$$28176m\ddot{l} + 2451mg = u \qquad (9)$$

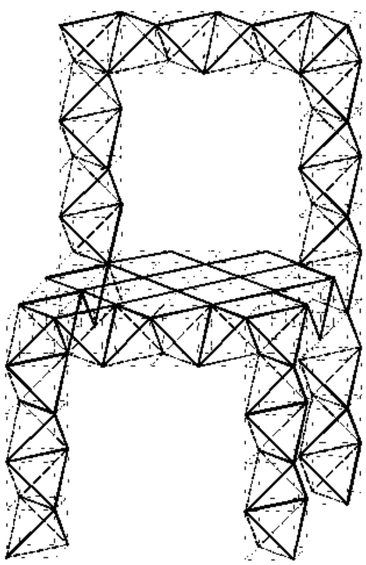

Fig.13: An alternative lattice that approximates a chair. An expanding tetrahedron is inscribed within every cube forming the skeleton of the chair.

The fabrication of these designs has a number of practical considerations. A typical problem with the construction of parallel prismatic joints is binding. In the lattice examples above, there are many parallel prismatic joints. If the system contains a certain amount of compliance, as one prismatic member is lengthened, the other parallel members may not lengthen as much. This often leads to an increase in the normal force in the non-driven member, resulting in further lag in the lengthening until binding occurs and the system seizes. By properly placing multiple actuators within the lattice that work in concert, this type of binding may be avoided.

7. Conclusion

The paper proposes a new approach for designing and constructing single degree-of-freedom expanding structures using single degree-of-freedom expanding polyhedral units. The polyhedral units use the geometry of regular polygons with edges replaced by prismatic joints. Using polyhedral units, a large number of useful lattices can be built to approximate a given three-dimensional shape. On actuation, the lattice can expand while retaining the exterior shape. The governing dynamic equations for cubic, tetrahedral, and octahedral lattices are developed and illustrated by an example of a chair that adjusts in size. We foresee applications of these principles to a large number of other designs.

Mass Matrix(m_{ijk}) j=0							
i / k	0	1	2	3	4	5	6
10	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
5	0	5	5	0	0	5	0
4	5	0	5	0	5	0	5
3	0	5	0	0	0	5	0
2	5	0	0	0	0	0	5
1	0	5	0	0	0	5	0
0	3	0	0	0	0	0	3

Table 2: The j=0 nodal masses for the chair shown in Fig. 13.

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